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Thin-film and curtain flows on the outside of a rotating horizontal cylinder

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We use the lubrication approximation to investigate the steady two-dimensional flow of a thin film of viscous fluid on the outside of a rigid circular cylinder that is rotating about its (horizontal) axis. Primarily we are concerned with the flow that ensues when fluid is supplied continuously as a 'curtain' from above the cylinder, so that it flows round the cylinder and eventually falls off near the bottom. This problem may be thought of as a 'hybrid' of the two classical problems studied by Nusselt (1916*a*,*b*) and Moffatt (1977), concerning, respectively, flow on a stationary cylinder with a prescribed supply flux, and flow on a rotating cylinder when the supply flux is zero. For all these problems there are indeterminacies in the steady lubrication solution; we present a variety of possible solutions, including both 'full-film' and 'partial-film' solutions, and solutions that involve smooth 'jumps' in the free-surface profile. We show, for example, that stagnation points can occur in the flow, that solutions exist that do not have top-to-bottom symmetry, that in curtain flows the curtain generally takes a characteristic 'buckled' shape, and that in full-film curtain flows there is always some fluid that is 'trapped' near the rotating cylinder, never escaping as part of the curtain that detaches at the bottom of the cylinder. Also we show that finite-thickness films involving jumps cannot occur in these coating flows (though they are known to occur in rimming flows).

1. Introduction

Applying a thin layer of fluid to a substrate is a process of great industrial importance, occurring in, for example, the electronics industry, the paint industry and the food industry. (See, for instance, the review by Ruschak 1985, the recent European Coating Symposium Proceedings (Gaskell, Savage & Summers 1996; Bourgin 1998), and the comprehensive volume edited by Kistler & Schweizer (1997).) Many different coating devices have been developed, and quite commonly these involve flow over rotating cylindrical rollers; it is flows of this general form that are analysed in the present work. Essentially two different arrangements are discussed. In one the fluid is considered to be supplied (at a prescribed flux) as a 'curtain' incident on a solid cylinder that is rotating about its horizontal axis; the curtain comes from above the highest point of the cylinder, so that the fluid flows round the cylinder and eventually falls off near the bottom. The other arrangement involves a fixed mass of fluid on a horizontal rotating cylinder, with the film thickness finite everywhere. We take the fluid to be Newtonian and incompressible, and we take the fluid film to be thin, so that the lubrication approximation to the Navier–Stokes equations and the

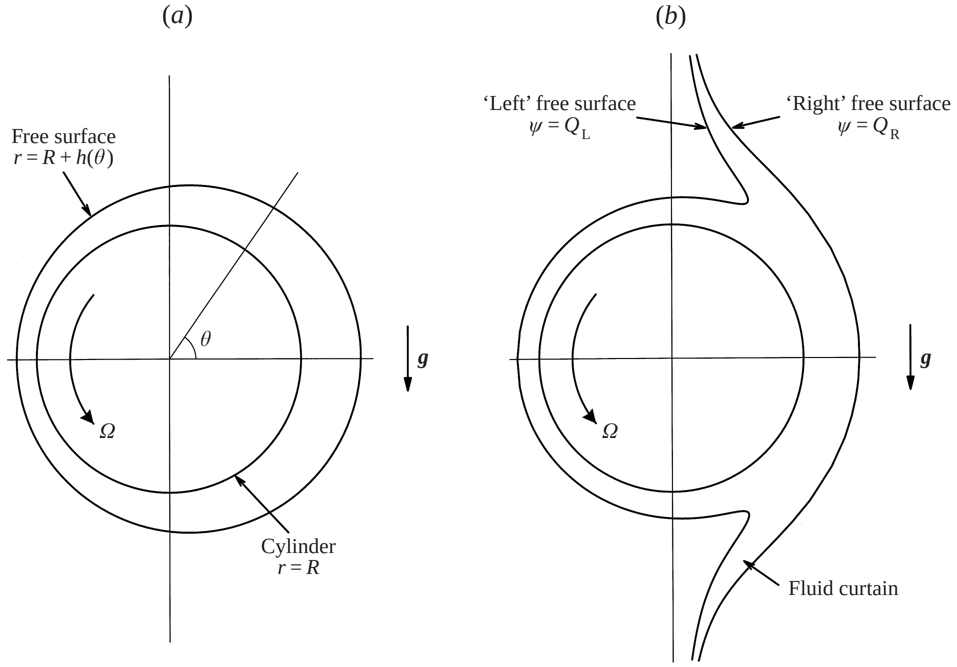

FIGURE 1. Examples of thin viscous film flows on a rotating cylinder. In (*a*) the film is of finite thickness everywhere; in (*b*) the fluid is supplied as a curtain from above.

boundary conditions is appropriate. Only steady, two-dimensional flow is considered, and surface-tension effects are neglected.

The curtain problem may be thought of as a 'hybrid' of the two well-known thin-film problems considered in the classical studies of Nusselt (1916*a,b*) and Moffatt (1977). Nusselt considered the case when there is a prescribed supply flux incident on a stationary cylinder; he showed, in particular, that the thickness of the film varies as $|\cos\theta|^{-1/3}$, where θ is the angle measured round the cylinder (as in figure 1). Thus the film thickness becomes infinite at the top ($\theta = \frac{1}{2}\pi$) and bottom ($\theta = -\frac{1}{2}\pi$) of the cylinder; these singularities may crudely be interpreted as representing the fluid falling onto and falling off the cylinder (though of course the lubrication approximation breaks down near $\theta = \pm\frac{1}{2}\pi$). Moffatt considered flow of a fixed mass of fluid on a rotating cylinder, the film thickness in this case being finite for all θ. In particular, he determined the maximum weight of fluid that can be kept on the cylinder for a given rotation rate.

For the hybrid problem we present a variety of possible profiles of the free surface of the fluid; we highlight a non-uniqueness in the solution (akin to those encountered in other thin-film flows of this sort), namely that prescribing the supply flux and the rotation rate is not sufficient to determine the solution completely. Thus extra conditions (such as, for example, the specification of the film thickness at some prescribed point) would be necessary to determine a unique solution.

In fact, it will emerge that several different types of solution are possible, and it is useful to establish some terminology immediately. A *full* film is one that wets all the cylinder, whereas a *partial* film is one that wets only part of it. Also a solution will be described as a *jump* solution if there is a rapid (but continuous) change in the film thickness at some point.

We deal here only with flow on the outside of a cylinder ('coating flow'). Other studies of such flows include those of Campanella & Cerro (1984) (who considered the case when the rotating cylinder is partially submerged in a bath of fluid), Preziosi & Joseph (1988) (who considered the axial variation of the flow), Hansen & Kelmanson (1994) (who performed a numerical study of Stokes flow on a cylinder), and Kelmanson (1995) (who extended Moffatt's thin-film analysis to include certain higher-order terms); all these papers deal only with 'non-curtain' films. Problems concerning flow on the inside of a cylinder ('rimming flow') have also been considered extensively, by, for example, Deiber & Cerro (1976), Ruschak & Scriven (1976), Orr & Scriven (1978), Preziosi & Joseph (1988), Johnson (1988), Wilson & Williams (1997) and O'Brien & Gath (1998).

2. Governing equations

We consider two-dimensional arrangements of the type indicated in figure 1, depicting examples of films of Newtonian fluid of constant density ρ and viscosity μ flowing on the exterior of a circular cylinder of radius R rotating in a counter-clockwise sense about its horizontal axis at uniform angular speed Ω (so that the circumferential speed is $U = R\Omega$). Figure 1(a) shows a case where the film is of finite non-zero thickness everywhere, and figure 1(b) shows a case in which fluid is supplied as a curtain; cases involving, for example, partial films are also included in the analysis.

Referred to polar coordinates r, θ with origin at the cylinder's axis and with θ measured counter-clockwise from the horizontal, we take the free surface of the fluid to be at $r = R + h(\theta)$, the film thickness being denoted by $h(\theta)$. We will consider only thin films for which $\delta \ll 1$, where $\delta := h_0/R$, with h_0 denoting a characteristic film thickness. Then the lubrication approximation leads to the equation

$$\nu u_{rr} = g\cos\theta \tag{1}$$

for the azimuthal velocity component $u(r,\theta)$, together with the boundary conditions

$$u = U \quad \text{on} \quad r = R, \tag{2}$$

$$u_r = 0 \quad \text{on} \quad r = R + h(\theta), \tag{3}$$

where g denotes gravitational acceleration, $\nu = \mu/\rho$, and suffixes denote partial differentiation. Thus

$$u = U - \frac{g\cos\theta}{2\nu}(2hy - y^2), \tag{4}$$

where y is a normal coordinate defined by

$$y = r - R \tag{5}$$

(so that $y \geqslant 0$ for flow on the exterior of the cylinder). The flux Q of fluid per unit axial length crossing a station $\theta =$ constant (in the direction of increasing θ) is

$$Q = \int_0^h u\,\mathrm{d}y = Uh - \frac{gh^3}{3\nu}\cos\theta. \tag{6}$$

As the flow is steady we have $\partial Q/\partial\theta = 0$, so that $Q(\theta)$ must be (piecewise) constant. The fact that h and u depend on θ only through $\cos\theta$ shows that the flow has top-to-bottom symmetry, at least when the film is full and does not involve a jump.

The stream function ψ, satisfying $u = \psi_y$, is given by

$$\psi = Uy - \frac{g\cos\theta}{6\nu}(3h - y)y^2, \tag{7}$$

if the normalization $\psi = 0$ on $r = R$ ($y = 0$) is used. On the free surface $y = h(\theta)$ we then have $\psi = Q$. Also the free-surface velocity $u_s(\theta) := u(R + h, \theta)$ is given by

$$u_s(\theta) = U - \frac{gh^2}{2\nu}\cos\theta. \tag{8}$$

In a situation of the type shown in figure 1(b) we suppose that a fluid 'curtain' (of prescribed flux $Q_S > 0$) falls onto the cylinder from above, and a corresponding curtain detaches from the cylinder near the bottom. If the two free surfaces of the curtains correspond to $\psi = Q_R$ and $\psi = Q_L$ (the suffixes R and L referring to right and left in the figures), then a global mass balance gives

$$Q_L - Q_R = Q_S. \tag{9}$$

3. Solution for a stationary cylinder ($U = 0$)

For the special case of a stationary cylinder ($U = 0$) there is no steady solution of the type shown in figure 1(a), but curtain flows of the type shown in figure 1(b) are possible. For such a flow we may write

$$Q(\theta) = \begin{cases} (1-k)Q_S & \text{if } \cos\theta < 0 \\ -kQ_S & \text{if } \cos\theta > 0, \end{cases} \tag{10}$$

where k is a constant satisfying $0 < k < 1$, so that $Q_L = (1-k)Q_S$ (> 0), $Q_R = -kQ_S$ (< 0), and (9) is satisfied. (The sign difference here arises because when $U = 0$ the flux is everywhere downwards, and so it is in the direction of increasing θ on the left of the cylinder but is in the direction of decreasing θ on the right.) The free-surface profile is then given from (6) by

$$h(\theta) = \left(\frac{-3\nu Q(\theta)}{g\cos\theta}\right)^{1/3}, \tag{11}$$

or, in non-dimensional form,

$$\tilde{h} = \left(\frac{3\left[k - \mathrm{H}(-\cos\theta)\right]}{\cos\theta}\right)^{1/3}, \tag{12}$$

where $\tilde{h} = (g/\nu Q_S)^{1/3}h$, and $\mathrm{H}(\cdot)$ denotes the Heaviside unit-step function. We note that this solution does not in general have left-to-right symmetry; however, the solution for a given k is the mirror image of the solution obtained by replacing k by $1-k$.

The constant k, which is a measure of the relative amount of fluid that goes round each side of the cylinder, is not determined by the present theory, so (12) represents a one-parameter family of possible (steady) solutions. Prescribing the supply flux Q_S alone is not sufficient to determine the solution uniquely; however, additional information, such as (for example) a specification of the film thickness at some prescribed point, would be sufficient to render a unique solution. This indeterminacy is akin to the well-known indeterminacy occurring in forward roll coating between

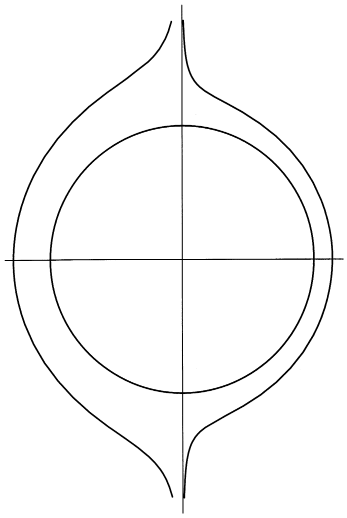

FIGURE 2. The solution (12) for thin-film flow round a stationary cylinder, drawn for the case $k = \frac{1}{9}$.

two rollers, when a fluid film 'splits', with some fluid attaching to one roller and some to the other; again lubrication theory is unable to predict how much goes round each, unless additional conditions are imposed. In drag-out problems of the type considered by Van Rossum (1958), Tuck (1983) and Tuck, Bentwich & Van der Hoek (1983), for example, there is again an indeterminacy, but of a somewhat different form. In these problems it is the flux of fluid carried up by the substrate that is unknown *a priori*, and there is a different solution for each value of this flux. Tuck (1983) provides a little historical background on this difficulty, saying that there is a 'varied and sometimes apparently contradictory literature on the problem'. Tuck (1983) and Tuck *et al.* (1983) quote a suggestion of Deryagin that in practice the film will take up a thickness that will maximize the flux. In the present problem, on the other hand, the flux Q_S is prescribed; it is the 'split' of this between Q_R and Q_L that has to be determined by alternative means.

Figure 2 shows an example of the solution (12), drawn for the case $k = \frac{1}{9}$ (the film thickness then being such that $h(\pi) = 2h(0)$). Nusselt's (1916*a,b*) familiar classical solution corresponds to the particular case $k = \frac{1}{2}$, so that $Q_L = -Q_R = \frac{1}{2}Q_S$ and hence

$$h(\theta) = \left(\frac{3\nu Q_S}{2g|\cos\theta|}\right)^{1/3}, \tag{13}$$

which has left-to-right symmetry.

4. Flow on a rotating cylinder ($U \neq 0$): qualitative features

4.1. *Solution structure*

From now on we consider the general case of a rotating cylinder (with $U > 0$). Here we mainly describe mathematical features of the solution, obtaining general results for all of the cases considered subsequently; in later sections we give more details of possible physical interpretations.

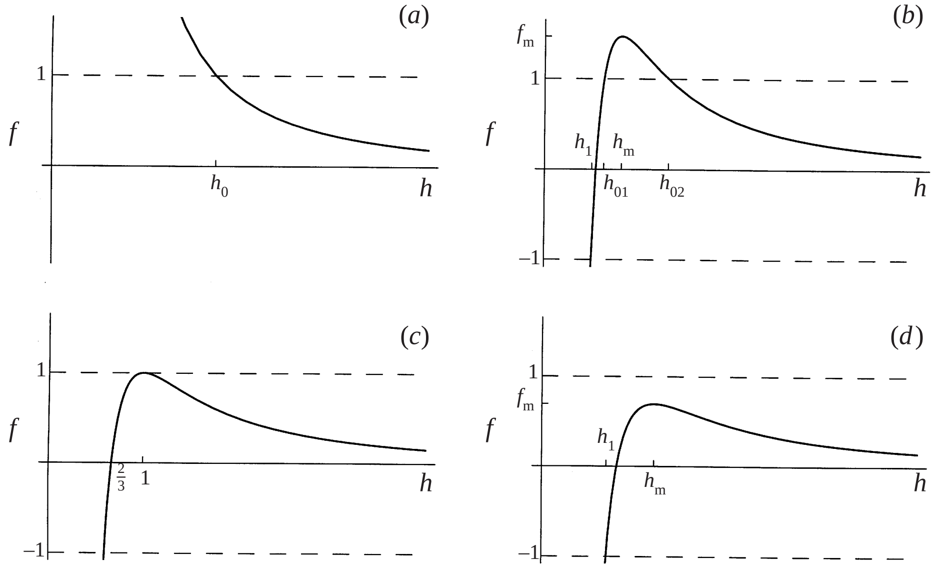

FIGURE 3. Typical plots of $f(h, Q)$ in equation (20) as a function of h, for different values of the flux Q: (a) $Q \leqslant 0$, (b) $0 < Q < \frac{2}{3}$, (c) $Q = \frac{2}{3}$, and (d) $Q > \frac{2}{3}$.

We non-dimensionalize using the scheme

$$y = Ly^*, \quad h = Lh^*, \quad u = Uu^*, \quad u_s = Uu_s^*, \quad \psi = UL\psi^*, \quad Q = ULQ^*, \tag{14}$$

where

$$L = (\nu U/g)^{1/2}. \tag{15}$$

The star superscripts will be dropped immediately. We thus have

$$u = 1 - \tfrac{1}{2}\cos\theta(2hy - y^2), \tag{16}$$

$$u_s(\theta) = 1 - \tfrac{1}{2}\cos\theta h^2, \tag{17}$$

$$\psi = y - \tfrac{1}{6}\cos\theta(3h - y)y^2, \tag{18}$$

and from (6) the free-surface profile $h(\theta)$ is given by

$$\cos\theta = f(h), \tag{19}$$

where

$$f(h, Q) = 3\left(\frac{h - Q}{h^3}\right). \tag{20}$$

(The argument Q in $f(h, Q)$ will be suppressed if no confusion can arise.) From (18)–(20) we can obtain the useful identity

$$\psi - Q = (h - y)\left[-1 + \tfrac{1}{6}\cos\theta(2h^2 + 2hy - y^2)\right]. \tag{21}$$

Typical plots of $f(h)$ in (20) for different values of Q are shown in figure 3, for the relevant domain $h > 0$; of course, by (19) only values of h such that $|f(h)| \leqslant 1$ are physically sensible. In all cases $f(h) \sim 3/h^2 \to 0^+$ as $h \to \infty$. For $Q \leqslant 0$ the function $f(h)$ decreases monotonically with h, satisfying $f(h) \sim -3Q/h^3 \to +\infty$ as $h \to 0$ for $Q < 0$ and satisfying $f(h) \sim 3/h^2 \to +\infty$ as $h \to 0$ for $Q = 0$. For $Q > 0$ the function

$f(h)$ has a maximum given by

$$f = f_{\mathrm{m}} = \frac{4}{9Q^2}, \quad \theta = \theta_{\mathrm{c}} = \cos^{-1} f_{\mathrm{m}}, \quad h = h_{\mathrm{m}} = \frac{3Q}{2} \tag{22}$$

(so the maxima for the different curves satisfy $f_{\mathrm{m}} = 1/h_{\mathrm{m}}^2$). Also $f(h) = 0$ when $h = Q$ (> 0), and $f(h) \sim -3Q/h^3 \to -\infty$ as $h \to 0$.

Depending on the value of Q, different branches of the above curves can be relevant; it is useful to distinguish five cases, which we label I–V, as follows. For $Q \leqslant 0$ the only relevant branch is given by

$$\text{I}: \quad h \geqslant h_0, \ 0 < f(h) \leqslant 1, \ |\theta| < \tfrac{1}{2}\pi \quad \text{for } Q \leqslant 0, \tag{23}$$

where $h_0 := h(0)$ (that is, h_0 is the unique real root of $f(h_0) = 1$). This branch represents the right-hand free surface in a curtain flow, with h extending from h_0 at $\theta = 0$ to infinity at $\theta = \pm\frac{1}{2}\pi$.

For $Q > 0$ the curve has, in effect, two branches in $h > 0$, one on each side of the maximum at $h = h_{\mathrm{m}}$; h remains finite on the branch in $h \leqslant h_{\mathrm{m}}$, but can become infinite on the branch in $h > h_{\mathrm{m}}$. We let $h_1 := h(\pi)$, that is, we let h_1 be the unique solution of $f(h_1) = -1$. For $0 < Q \leqslant \frac{2}{3}$ there are two values h_0 that satisfy $f(h_0) = 1$; let these be h_{01} and h_{02}, say, with $h_{01} \leqslant h_{\mathrm{m}} \leqslant h_{02} < \sqrt{3}$. Then the two relevant branches are given by

$$\left.\begin{array}{ll} \text{II}: & h_1 \leqslant h \leqslant h_{01}, \ -1 \leqslant f(h) \leqslant 1, \ \text{all } \theta \\ \text{III}: & h \geqslant h_{02}, \ 0 < f(h) \leqslant 1, \ |\theta| < \tfrac{1}{2}\pi \end{array}\right\} \text{for } 0 < Q \leqslant \tfrac{2}{3}. \tag{24}$$

For $Q > \frac{2}{3}$ there is no value h_0 that satisfies $f(h_0) = 1$ (see figure 3); the two relevant branches are given by

$$\left.\begin{array}{ll} \text{IV}: & h_1 \leqslant h \leqslant h_{\mathrm{m}}, \ -1 \leqslant f(h) \leqslant f_{\mathrm{m}}, \ \theta_{\mathrm{c}} \leqslant |\theta| \leqslant \pi \\ \text{V}: & h \geqslant h_{\mathrm{m}}, \ 0 < f(h) \leqslant f_{\mathrm{m}}, \ \theta_{\mathrm{c}} \leqslant |\theta| < \tfrac{1}{2}\pi \end{array}\right\} \text{for } Q > \tfrac{2}{3}, \tag{25}$$

with θ_{c} as in (22). Branch II corresponds to Moffatt's (1977) solution, involving a finite-thickness film encircling the cylinder. Branch III is somewhat similar to branch I. Branch IV corresponds to the free surface on the left of the cylinder in a curtain flow, while branch V corresponds to the left free surface of the curtain itself.

Figure 4 shows schematically how these different branches are to be interpreted physically (but note that not all of the branches shown can occur together in a given physical context). The explicit form for $h(\theta)$ on each of the branches is given in the Appendix. The point A in figure 4 separates branch IV and branch V on the upper curtain; this point is at $\theta = \theta_{\mathrm{c}}$, where

$$\theta_{\mathrm{c}} = \cos^{-1}\left(\frac{4}{9Q_{\mathrm{L}}^2}\right), \tag{26}$$

which is obtained from (22) with $Q = Q_{\mathrm{L}}$.

The case $Q = \frac{2}{3}$ turns out to be 'critical' because its maximum is $f_{\mathrm{m}} = 1$, corresponding to $\cos\theta = 1$ and $h = 1$. Previous studies (for example, Moffatt 1977; Johnson 1988; Preziosi & Joseph 1988) have been concerned mainly (or exclusively) with solutions for which $0 < Q \leqslant \frac{2}{3}$ (in the present notation), which represent films of finite thickness, on the inside or the outside of a cylinder. The present study includes

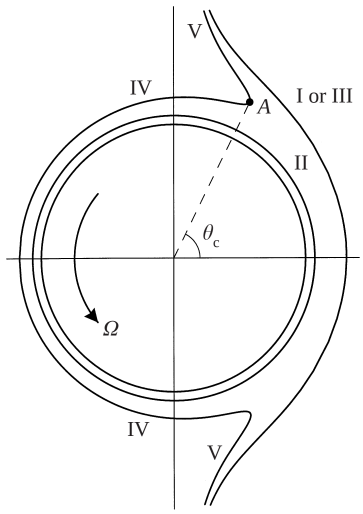

FIGURE 4. Schematic representation of the branches I,...,V of the solution $h(\theta)$ of (19) and (20). The point A separates branch IV and branch V on the upper curtain. (Note that in a given physical context not all of the branches shown can be relevant.)

solutions for values of Q outside this interval; these represent flows involving fluid curtains.

Since by definition $Q_R = Q(0)$ and $Q_L = Q(\pi)$ we may use (19) and (20) to obtain

$$Q_R = h_0 - \tfrac{1}{3}h_0^3, \quad Q_L = h_1 + \tfrac{1}{3}h_1^3, \tag{27}$$

where again $h_0 = h(0)$ and $h_1 = h(\pi)$, the relevant branch of solutions being such that $Q_R \leqslant \frac{2}{3}$, $h_0 \geqslant 1$; clearly Q_L must be positive. We note from (19) and (20) that for $Q_R < \frac{2}{3}$ the free-surface shape near $\theta = 0$ is roughly parabolic in θ, with

$$h(\theta) = h_0 + \frac{h_0^3}{6(h_0^2-1)}\theta^2 + \frac{h_0^3(3h_0^4 - 4h_0^2 - 1)}{72(h_0^2-1)^3}\theta^4 + O(\theta^6) \quad \text{as } \theta \to 0, \tag{28}$$

while in the special case $Q_R = \frac{2}{3}$ $(h_0 = 1)$ we have

$$h(\theta) = 1 \pm \frac{\theta}{\sqrt{6}} + O(\theta^2) \quad \text{as } \theta \to 0. \tag{29}$$

The latter may be interpreted as representing either a locally linear free surface (for a 'jump' solution), or a free surface with a corner (for a 'non-jump' solution).

Equations (19) and (20) predict that, for a curtain solution, far from the cylinder the two free surfaces of the oncoming and detaching curtains have the forms

$$h = \left(\frac{3}{\frac{1}{2}\pi - |\theta|}\right)^{1/2} - \frac{Q}{2} + O\left(\left(\tfrac{1}{2}\pi - |\theta|\right)^{1/2}\right) \quad \text{as } |\theta| \to \tfrac{1}{2}\pi^-, \tag{30}$$

with $Q = Q_R$ or $Q = Q_L$; however, the fact that $h \to \infty$ and $|\partial h/\partial\theta| \to \infty$ renders the lubrication approximation invalid there (just as it does in Nusselt's solution in the case $U = 0$).

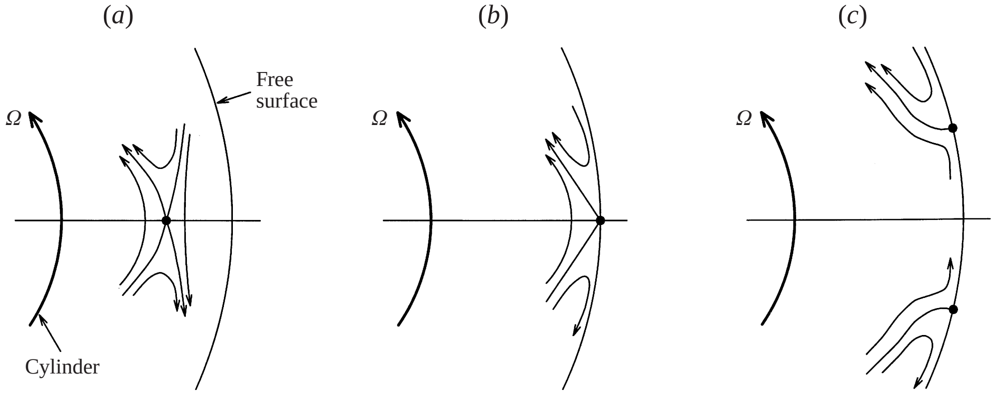

FIGURE 5. Sketch of the forms of the streamlines near the stagnation points in the cases (*a*) $Q_R < \frac{1}{3}\sqrt{2}$, (*b*) $Q_R = \frac{1}{3}\sqrt{2}$, and (*c*) $\frac{1}{3}\sqrt{2} < Q_R < \frac{2}{3}$.

4.2. *Stagnation points*

Stagnation points occur where $\psi_y = 0$ and $\psi_\theta = 0$; they are saddle points of the function $\psi(y, \theta)$ $\big($that is, $\psi_{yy}\psi_{\theta\theta} - \psi_{y\theta}^2 < 0$ there, at least for $Q_R \neq \frac{1}{3}\sqrt{2}\big)$, so the streamlines (the contours of ψ) are locally open curves.

If $Q_R < \frac{1}{3}\sqrt{2}$ then for a branch-I or a branch-III solution there is a single 'internal' stagnation point, at $\theta = 0$ and $y = y_s < h_0$, where

$$y_s = h_0 - \sqrt{h_0^2 - 2}, \tag{31}$$

with $h_0 > \sqrt{3}$ on branch I and with $h_0 = h_{02} > \sqrt{2}$ on branch III. The stagnation streamline is given by $\psi = \psi_s$, where

$$\psi_s = y_s - \tfrac{1}{6}(3h_0 - y_s)y_s^2, \tag{32}$$

which with (31) may be written

$$\psi_s = h_0 - \tfrac{1}{3}h_0^3 + \tfrac{1}{3}(h_0^2 - 2)^{3/2}. \tag{33}$$

One can show that near this stagnation point (that is, for $y \to y_s$ and $\theta \to 0$)

$$\psi = \psi_s + \frac{\sqrt{h_0^2 - 2}}{2}\left[\frac{\theta^2}{3(h_0^2 - 1)} - (y - y_s)^2\right] + o\big(\theta^2, (y - y_s)^2\big); \tag{34}$$

thus the streamlines are locally hyperbolae, and the stagnation streamline is given by

$$y = y_s \pm \frac{1}{\sqrt{3(h_0^2 - 1)}}\theta + \frac{3h_0^3 y_s - 9h_0^2 + 8}{18(h_0^2 - 1)(y_s - h_0)}\theta^2 + o(\theta^2) \text{ as } \theta \to 0, \tag{35}$$

so it comprises locally a pair of straight lines, at first order in θ. (Only two branches of the curve $\psi = \psi_s$ are relevant, the third being outside the fluid domain, in general.) An example of a flow of this type is shown in figure 5(*a*).

For a branch-III profile with $Q_R = \frac{1}{3}\sqrt{2}$ there is one stagnation point, which occurs on the free surface at $\theta = 0$, $y = y_s = h_0 = \sqrt{2}$. Near this point the stagnation streamline comprises the lines

$$y = \sqrt{2} - |\theta| + O(\theta^2) \quad \text{as } \theta \to 0 \tag{36}$$

and the free surface $y = h$, which is of the form

$$h = \sqrt{2}\left(1 + \tfrac{1}{3}\theta^2 + \tfrac{1}{12}\theta^4\right) + O(\theta^6) \quad \text{as } \theta \to 0. \tag{37}$$

An example of a flow of this type is shown in figure 5(*b*).

For a branch-III profile with $Q_R > \frac{1}{3}\sqrt{2}$ there are stagnation points on the free surface at $y = h = 3Q_R$ ($> \sqrt{2}$) and $\theta = \pm\theta_s$, with

$$\theta_s = \cos^{-1}\left(\frac{2}{9Q_R^2}\right). \tag{38}$$

The stagnation streamline is given by $\psi = Q_R$, and from (21) with $Q = Q_R$ we see that it comprises the free surface $y = h(\theta)$ and the curve given by

$$\frac{y}{h} = 1 - \left(3 - \frac{6}{h^2\cos\theta}\right)^{1/2} \quad \text{for } 2 \leqslant h^2\cos\theta \leqslant 3, \tag{39}$$

or equivalently

$$\frac{y}{h} = 1 - \left(\frac{h - 3Q_R}{h - Q_R}\right)^{1/2} \quad \text{for } \sqrt{2} < 3Q_R \leqslant h. \tag{40}$$

In the vicinity of the 'upper' stagnation point at $y = h = 3Q_R$, $\theta = \theta_s$, the free surface has the form

$$h(\theta) = 3Q_R + 9Q_R^3\sin\theta_s(\theta - \theta_s) + \tfrac{1}{3}Q_R(81Q_R^4 - 1)(\theta - \theta_s)^2 + O(\theta - \theta_s)^3, \tag{41}$$

and the branch of the stagnation streamline given by (40) has a semi-parabolic form:

$$y = 3Q_R - 9Q_R^2(\tfrac{1}{2}\sin\theta_s)^{1/2}(\theta - \theta_s)^{1/2} + 9Q_R^3\sin\theta_s(\theta - \theta_s) + O(\theta - \theta_s)^{3/2}, \tag{42}$$

the latter valid in $\theta \geqslant \theta_s$. The corresponding results for the 'lower' stagnation point at $\theta = -\theta_s$ are similar. An example of a flow of this type is shown in figure 5(*c*).

5. Films supplied by a fluid curtain

In this section we consider the general case $Q_S \neq 0$, that is, a non-zero (prescribed) flux of fluid is falling onto and off the cylinder continuously, as shown in figure 1(*b*).

5.1. *Full films*

For a full film we use two of the $f(h)$ curves to give the two sides of the curtain, one curve corresponding to $Q = Q_R$ and the other to $Q = Q_L$ (with $Q_L - Q_R = Q_S$). However, we must restrict Q_R by $Q_R \leqslant \frac{2}{3}$, for otherwise the right side of the curtain could not attain $\theta = 0$; similarly we need $Q_L \geqslant \frac{2}{3}$, for otherwise the left side of the curtain could not attain $\theta = \pi$. Thus if we write

$$Q_L = \tfrac{2}{3} + (1 - k)Q_S, \quad Q_R = \tfrac{2}{3} - kQ_S, \tag{43}$$

where k is a constant, then the only cases that give physically sensible solutions correspond to branches of curves for which $0 \leqslant k \leqslant 1$.† The parameter k is a measure of the relative fluxes of fluid round the two sides of the rotating cylinder. Again the present theory does not determine a unique value for k, and additional information would be required to render a unique solution.

† In the case of 'non-jump' solutions the values $k = 0$ and $k = 1$ lead to free surfaces that have 'corners', because $Q_R = \frac{2}{3}$ and $Q_L = \frac{2}{3}$ in these cases, respectively; cf. (29).

The free surface $\psi = Q_L$ comprises the two branches IV and V, with $Q = Q_L$. If $0 < Q_R \leqslant \frac{2}{3}$ then the free surface $\psi = Q_R$ comprises branch III (with $Q = Q_R$), whereas if $Q_R \leqslant 0$ then it comprises branch I (with $Q = Q_R$). Note that $Q_L > 0$, so the overall flow on the left side of the cylinder is always downwards (that is, in the direction of increasing θ), but Q_R may be positive or negative, so the flow on the right side may overall be upwards (with $1 < h_0 < \sqrt{3}$) or downwards (with $h_0 > \sqrt{3}$). On the right of the flow the film thickness is a minimum at $\theta = 0$.

Examples of free-surface profiles of this type are shown in figure 6 for various values of the flux parameters Q_R and Q_L (or equivalently Q_S and k). Figures 6(*a*) and 6(*b*) give examples with $Q_S = 2$ (and $k = \frac{23}{24}$ and $k = \frac{1}{12}$, respectively), while figures 6(*c*) and 6(*d*) give examples with $Q_S = 10$ (and $k = \frac{29}{30}$ and $k = \frac{2}{3}$, respectively). In fact figure 1(*b*) also gives an example of such a profile, in the case $Q_R = -1$ and $Q_L = 1$, so that $Q_S = 2$ and $k = \frac{5}{6}$. The effect of varying k is illustrated here: figures 1(*b*), 6(*a*) and 6(*b*) have the same value of the supply flux Q_S, but are distinguished by their different values of k.

It is seen in all these flows that, unlike in the Nusselt case, the curtain takes a characteristic 'buckled' shape, due to the rotation of the cylinder; and according to the present theory, this buckling is always in a direction opposite to that of the local cylinder motion, that is, to the right in our pictures. The authors are not aware of any reports of experimental or computational work directly relevant to this sort of flow, so at this stage it is difficult to assess how realistic this prediction is.

Clearly regions of rather high curvature are predicted by this solution; in reality surface-tension effects (which are ignored in the present analysis) could be significant in these regions, and would tend to smooth out the free surface locally.

The part of the free surface $y = h(\theta)$ determined by $Q = Q_R$ is well defined for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, but, because of the 'buckled' shape of the curtain, the part determined by $\psi = Q_L$ is multivalued in the intervals $\theta_c \leqslant |\theta| < \frac{1}{2}\pi$, where again $\theta = \theta_c$ is the coordinate of the point labelled A in figure 4, defined in (26). Although this is apparently reasonable from a physical point of view, our solution cannot represent it in a consistent way, since the analysis here assumes that, for any θ, the space $0 < y < h$ is occupied fully by fluid – but this is not the case for $\theta_c \leqslant |\theta| < \frac{1}{2}\pi$, there being three different values of h involved at any θ in this interval: one for the free surface near the cylinder (the *attached* film), one for the left side of the curtain, and one for the right side.† Thus there is a discontinuity on $\theta = \theta_c$, and the part of the overall solution obtained using $Q = Q_R$ (branch I or III) is strictly valid only for $|\theta| < \theta_c$, while only branch IV can be used for the part of the solution with $Q = Q_L$. At $\theta = \theta_c$ we have $h = \frac{3}{2}Q_L$, and the free surface near this point is roughly parabolic, with

$$h = \tfrac{3}{2}Q_L - \tfrac{3}{4}Q_L^2(3\sin\theta_c)^{1/2}(\theta-\theta_c)^{1/2} + O(\theta-\theta_c). \tag{44}$$

There is a stagnation point on the part of the free surface $\psi = Q_L$, but it is at $h = 3Q_L$, $\theta = \cos^{-1}(2/9Q_L^2)$ $(= \theta_1$, say), which is above the point A, in $\theta > \theta_c$, and so is not relevant; similarly for a stagnation point at $\theta = -\theta_1$.

Incidentally, one might have anticipated that the free-surface profile when U is

† One might ask, for example, what is the velocity u at a given point in this θ interval. The answer is that there could be one, two or three values of u, depending on where the point is and what choice is made for h. Indeed two values of u are predicted even for points in the space outside the curtain, on its left side! The stated restrictions resolve this dilemma, but raise the questions of what profile the curtain would really have, and how it would match onto the film into which it plunges.

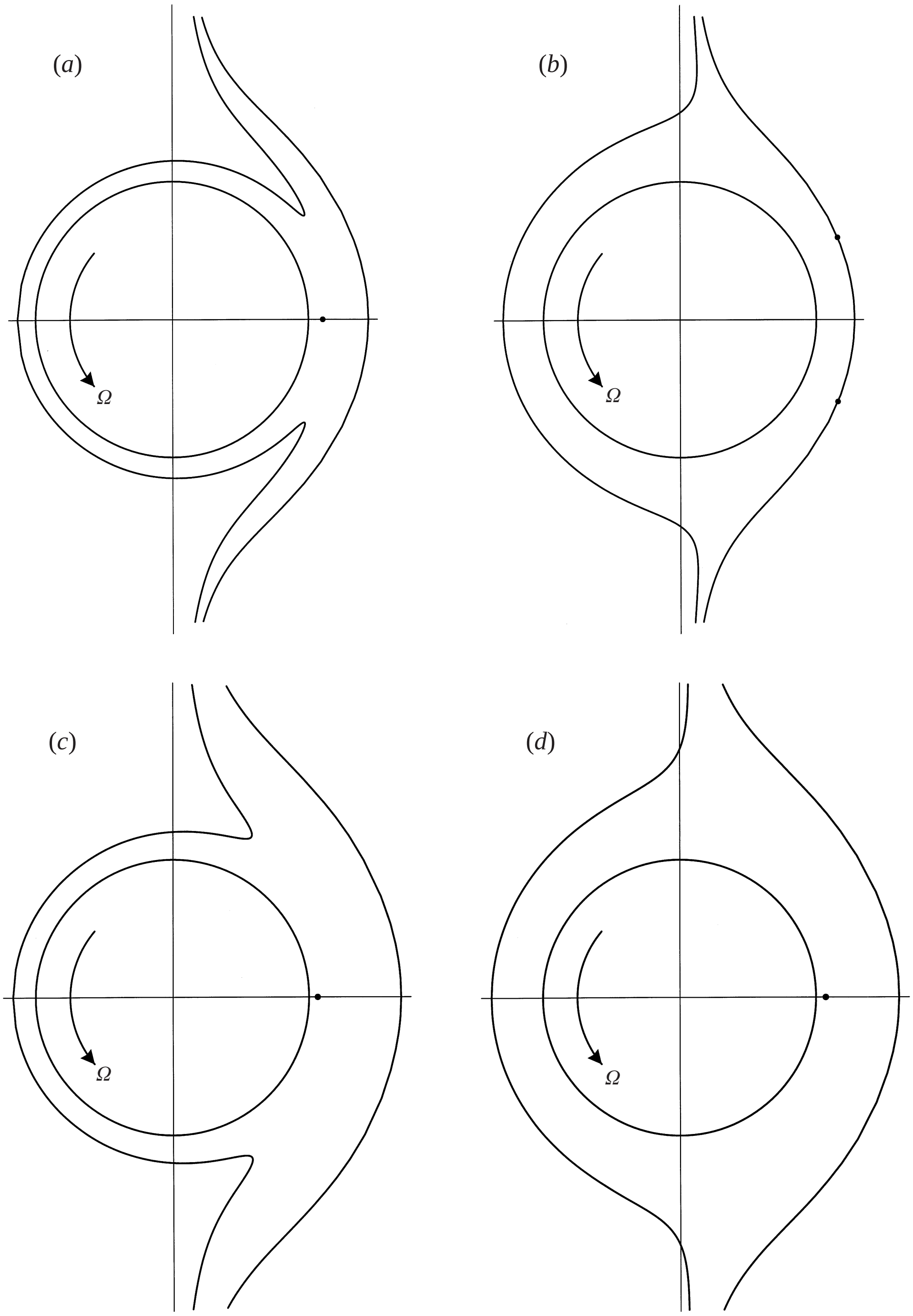

FIGURE 6. Some examples of curtain solutions, drawn for the cases (*a*) $Q_R = -\frac{5}{4}$, $Q_L = \frac{3}{4}$, $Q_S = 2$, $k = \frac{23}{24}$, (*b*) $Q_R = \frac{1}{2}$, $Q_L = \frac{5}{2}$, $Q_S = 2$, $k = \frac{1}{12}$, (*c*) $Q_R = -9$, $Q_L = 1$, $Q_S = 10$, $k = \frac{29}{30}$, and (*d*) $Q_R = -6$, $Q_L = 4$, $Q_S = 10$, $k = \frac{2}{3}$. The dots indicate the positions of stagnation points.

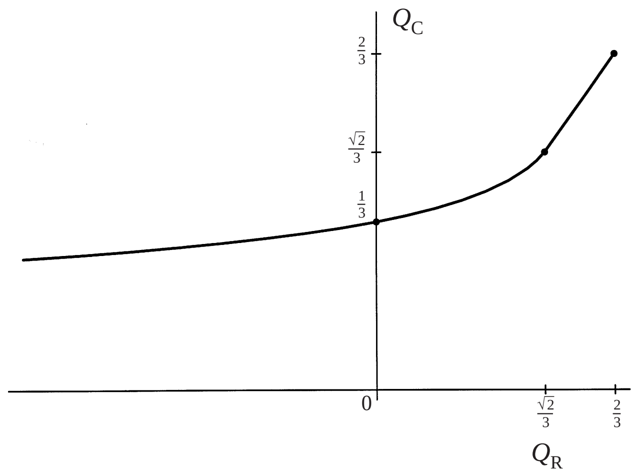

FIGURE 7. Plot of the circulation flux Q_C defined in (45) as a function of the flux parameter Q_R (which satisfies $Q_R \leqslant \frac{2}{3}$).

small would differ from that in the Nusselt case with $U = 0$ only in a region near the rotating cylinder, but that far from the cylinder the curtains in the two cases would look somewhat similar. In fact according to the present solution the opposite is the case, in the sense that the two profiles look rather similar near the cylinder, whereas the oncoming curtains are different: when $U = 0$ the curtain can be symmetric about $\theta = \frac{1}{2}\pi$, with $\theta \to \frac{1}{2}\pi^{\pm}$ on its two sides (cf. (13)), but when $U \neq 0$ it lies entirely in $\theta < \frac{1}{2}\pi$ sufficiently far from the cylinder, with $\theta \to \frac{1}{2}\pi^{-}$ on *both* sides of the curtain (cf. (30)). Essentially the asymmetric buckling of the film occurs however slow the cylinder's rotation may be – and the slower the rotation, the further away from the cylinder the buckling occurs.

A feature of these flows that is potentially of practical importance is that for any Q_R ($\leqslant \frac{2}{3}$) there is always some fluid that is 'trapped' near the rotating cylinder, circulating forever with the cylinder and never escaping as part of the curtain detaching at the bottom. Denoting the flux of this circulating fluid by Q_C we have $Q_C = \psi_s$ for $Q_R \leqslant \frac{1}{3}\sqrt{2}$ and $Q_C = Q_R$ for $Q_R \geqslant \frac{1}{3}\sqrt{2}$; thus Q_C is given parametrically as a function of Q_R by

$$Q_C = \begin{cases} h_0 - \frac{1}{3}h_0^3 + \frac{1}{3}(h_0^2 - 2)^{3/2} & \text{if } Q_R \leqslant \frac{1}{3}\sqrt{2},\ h_0 \geqslant \sqrt{2} \\ h_0 - \frac{1}{3}h_0^3 & \text{if } \frac{1}{3}\sqrt{2} \leqslant Q_R \leqslant \frac{2}{3},\ 1 < h_0 \leqslant \sqrt{2}, \end{cases} \tag{45}$$

with Q_R and h_0 related as in (27). A plot of Q_C as a function of Q_R is shown in figure 7. We have $Q_C = \frac{2}{3}$ when $Q_R = \frac{2}{3}$, $Q_C = \frac{1}{3}\sqrt{2}$ when $Q_R = \frac{1}{3}\sqrt{2}$, $Q_C \sim \frac{1}{3} + \left(1 - \frac{1}{2}\sqrt{3}\right)Q_R$ as $Q_R \to 0$, and $Q_C \sim \frac{1}{2}(-3Q_R)^{-1/3}$ as $Q_R \to -\infty$. Note also that

$$\tfrac{2}{3} - Q_S \leqslant Q_R \leqslant Q_C \leqslant \tfrac{2}{3} \leqslant Q_L \leqslant \tfrac{2}{3} + Q_S. \tag{46}$$

Note that in the cases shown in figures 1(*b*), 6(*a*) and 6(*b*), for example, Q_C takes different values (namely $Q_C \simeq 0.26$, $Q_C \simeq 0.25$ and $Q_C = \frac{1}{2}$, respectively), even though the supply flux $Q_S = 2$ is the same in each. For $Q_R \geqslant \frac{1}{3}\sqrt{2}$ ($h_0 \leqslant \sqrt{2}$) all the fluid in the oncoming curtain goes round the left side of the cylinder and into the detaching curtain at the bottom, whereas for $Q_R < \frac{1}{3}\sqrt{2}$ ($h_0 > \sqrt{2}$) some fluid goes round the left side and some round the right side. The surface velocity u_s at $\theta = 0$, namely

$u_s(0) = 1 - \frac{1}{2}h_0^2$, will correspondingly be upwards or downwards (though, of course, the fluid velocity at the cylinder $y = 0$ must always be upwards at $\theta = 0$, by the no-slip condition). Somewhat similar recirculating regions of 'trapped' fluid can occur in many other thin-film flows, including flow in a slider bearing (see, for example, Tuck & Bentwich 1983), flow in a slot coater (Durst & Wagner 1997), and flow in a roll coater (Coyle 1997; Gaskell & Savage 1997).

Lastly we note that near $\theta = \frac{1}{2}\pi$ the free surface $\psi = Q_L$ with h finite has the form

$$h = Q_L + \tfrac{1}{3}Q_L^3(\tfrac{1}{2}\pi - \theta) + O\big((\tfrac{1}{2}\pi - \theta)^2\big); \tag{47}$$

more generally a streamline $\psi =$ constant in the 'attached' film has the form

$$y = \psi + \tfrac{1}{6}\psi^2(3Q_L - \psi)(\tfrac{1}{2}\pi - \theta) + O\big((\tfrac{1}{2}\pi - \theta)^2\big) \tag{48}$$

near $\theta = \frac{1}{2}\pi$.

5.2. *Partial films*

Johnson (1988) found solutions for rimming flows in which the fluid covers only part of the cylinder (on the right), with the rest of the cylinder remaining dry (see his figures 1(*b*) and 8); we now show that analogous partial-film solutions can occur in curtain flows, with the film covering only part of the cylinder, while other parts (on the right) remain dry. In this case $Q = 0$, so that equation (19) yields the solution $h(\theta) = 0$ for a dry part of the cylinder, as expected, while for a wetted part it yields the branch-I solution

$$h(\theta) = \left(\frac{3}{\cos\theta}\right)^{1/2} \tag{49}$$

(which can be valid only if $|\theta| < \frac{1}{2}\pi$, i.e. on the right-hand side). Evidently (49) cannot satisfy the required condition that $h = 0$ at an edge $\theta = \theta_e$ of the film, which is therefore a region of non-uniformity of the present solution. However, near $\theta = \theta_e$ there is a 'transition solution' in which $\mathrm{d}h/\mathrm{d}\theta$ is large. As Johnson shows, in this transition region it is necessary to retain in the lubrication analysis a term proportional to $\delta(\mathrm{d}h/\mathrm{d}\theta)$, to arrive at

$$Q = h - \tfrac{1}{3}h^3(\cos\theta + \delta h_\theta \sin\theta), \tag{50}$$

so that equation (19) is replaced by

$$\delta \sin\theta\, h_\theta = f(h, Q) - \cos\theta, \tag{51}$$

with $f(h, Q)$ as in (20). With $Q = 0$ we now set $\theta = \theta_e + \delta\xi$ in (51); at leading order in δ we obtain

$$\delta \sin\theta_e\, h_\theta = f(h, 0) - \cos\theta_e, \tag{52}$$

to be solved subject to the boundary condition $h = 0$ at $\xi = 0$. This leads to the implicit solution

$$\frac{\theta - \theta_e}{\delta} = \frac{h_e^3 \sin\theta_e}{3}\left(\tanh^{-1}\frac{h}{h_e} - \frac{h}{h_e}\right), \qquad h_e = \left(\frac{3}{\cos\theta_e}\right)^{1/2}, \tag{53}$$

valid for any θ_e satisfying $|\theta_e| < \frac{1}{2}\pi$. This solution gives $h \to (3/\cos\theta_e)^{1/2}$ as $\mathrm{sgn}(\theta_e)(\theta - \theta_e)/\delta \to \infty$, as required for matching with (49). Also for $\theta \to \theta_e$ it gives $h \sim \big(9(\theta - \theta_e)/\delta \sin\theta_e\big)^{1/3}$, so that $h \to 0$ and $|h_\theta| \to \infty$ as $\theta \to \theta_e$ (i.e. the 'contact angle' at θ_e is $\frac{1}{2}\pi$). Thus (53) gives a suitable smooth but rapid transition from (49) down to $h = 0$ at $\theta = \theta_e$.

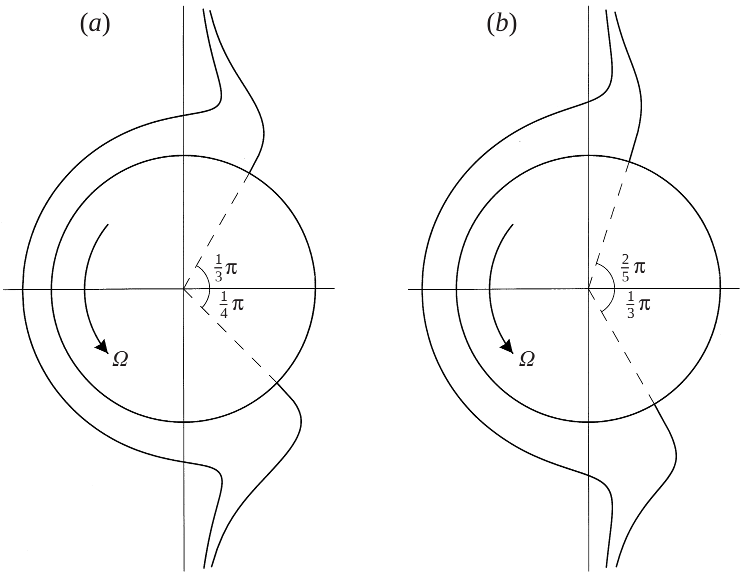

FIGURE 8. Two examples of curtain flows involving partial films, drawn for the cases (*a*) $Q_R = 0$, $Q_L = Q_S = \frac{3}{2}$, $k = \frac{4}{9}$, $\delta = 0.1$, $\theta_{e1} = -\frac{1}{4}\pi$, $\theta_{e2} = \frac{1}{3}\pi$, and (*b*) $Q_R = 0$, $Q_L = Q_S = 2$, $k = \frac{1}{3}$, $\delta = 0.025$, $\theta_{e1} = -\frac{1}{3}\pi$, $\theta_{e2} = \frac{2}{5}\pi$.

A uniformly valid composite expansion for $h(\theta)$ $\big($based on the one-term 'inner' and 'outer' expansions (53) and (49)$\big)$ is obtained straightforwardly. For example, for a partial film occupying $\theta \geqslant \theta_e$ we have

$$h_{\mathrm{comp}}(\theta) = \left(\frac{3}{\cos\theta}\right)^{1/2} + h_e\big[H(\eta) - 1\big], \quad \eta = \frac{3(\theta - \theta_e)}{\delta h_e^3 \sin\theta_e}, \tag{54}$$

with the function $H(\eta)$ defined by

$$\tanh^{-1} H - H = \eta. \tag{55}$$

In practice there must be two edge positions, $\theta = \theta_{e1}$ and $\theta = \theta_{e2}$ say, with $-\frac{1}{2}\pi < \theta_{e1} < 0 < \theta_{e2} < \frac{1}{2}\pi$, the dry portion of the cylinder occupying the interval $\theta_{e1} < \theta < \theta_{e2}$. The indeterminacy in the solution mentioned above is compounded here in that the edge positions θ_{e1} and θ_{e2} are apparently otherwise arbitrary; in particular the two edges need not be symmetrically placed about $\theta = 0$, that is, such a film need not have top-to-bottom symmetry. Examples of solutions of this sort are shown in figure 8(*a*) (with $\theta_{e1} = -\frac{1}{4}\pi$, $\theta_{e2} = \frac{1}{3}\pi$, $\delta = 0.1$, $Q_L = Q_S = \frac{3}{2}$ and $k = \frac{4}{9}$) and figure 8(*b*) (with $\theta_{e1} = -\frac{1}{3}\pi$, $\theta_{e2} = \frac{2}{5}\pi$, $\delta = 0.025$, $Q_L = Q_S = 2$ and $k = \frac{1}{3}$). Note that the supply flux $Q_S = 2$ in figure 8(*b*) is the same as that in figures 1(*b*), 6(*a*) and 6(*b*).

As is clear from figure 8, regions of rather high curvature are again predicted by this solution; surface-tension effects might smooth these out locally.

5.3. *Jump solutions*

As Johnson (1988) found for rimming flows, solutions involving 'jumps' from one branch of an $f(h)$-curve to another (that is, from one side of the maximum $h = h_m$ to the other) are also possible. Such a jump (a non-uniformity in the lubrication solution) would again represent a sudden change in the thickness of the film near some position $\theta = \theta_j$; it can occur only on the right of the flow. Physically we would

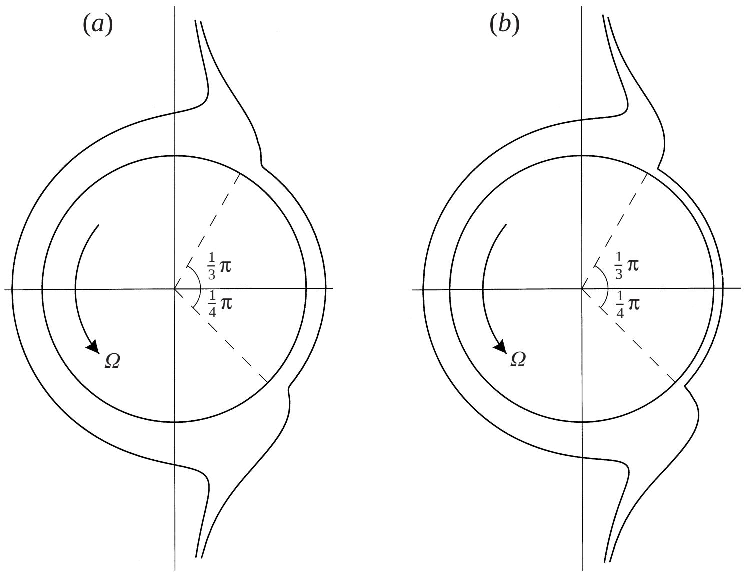

FIGURE 9. Two examples of jump-solution curtain flows, drawn for the cases (a) $Q_R = \frac{3}{5}$, $Q_L = \frac{8}{5}$, $k = \frac{1}{15}$, and (b) $Q_R = \frac{1}{3}$, $Q_L = \frac{4}{3}$, $k = \frac{1}{3}$, with $\theta_{j1} = -\frac{1}{4}\pi$, $\theta_{j2} = \frac{1}{3}\pi$, $Q_S = 1$ and $\delta = 0.1$ in both cases.

expect this change in profile to be rapid but smooth, akin to the solution found above for the region near an edge of a partial film. In the transition region the gradient of h will become large, so a term in δh_θ must again be retained in the lubrication analysis, leading again to (51). Then with $\theta = \theta_j + \delta\xi$ in (51), the equation for $h(\theta)$ that emerges at leading order in δ is

$$\delta \sin\theta_j \, h_\theta = f(h, Q) - \cos\theta_j, \tag{56}$$

which is to be solved subject to boundary conditions of the form

$$h \to h_- \quad \text{as} \quad \frac{\theta-\theta_j}{\delta} \to -\infty, \qquad h \to h_+ \quad \text{as} \quad \frac{\theta-\theta_j}{\delta} \to +\infty, \tag{57}$$

where h_- and h_+ denote the film thicknesses on either side of the transition at $\theta = \theta_j$. This problem can be solved analytically, but the (implicit) solution is rather unwieldly, so instead it was solved numerically, and a uniformly valid composite expansion was then constructed. As in the partial-film case, there must be two jumps of this sort, at $\theta = \theta_{j1}$ and $\theta = \theta_{j2}$, say, with $-\frac{1}{2}\pi < \theta_{j1} < 0 < \theta_{j2} < \frac{1}{2}\pi$; again the film need not have top-to-bottom symmetry. Essentially this profile comprises the solution from branch II (that is, Moffatt's solution) for $\theta_{j1} < \theta < \theta_{j2}$, and the solution from branch III for $\theta < \theta_{j1}$ and $\theta > \theta_{j2}$, connected by transition solutions at $\theta = \theta_{j1}$ and $\theta = \theta_{j2}$. Examples of profiles of this sort are shown in figure 9, drawn for the case $Q_S = 1$, $\delta = 0.1$, $\theta_{j1} = -\frac{1}{4}\pi$ and $\theta_{j2} = \frac{1}{3}\pi$, with $Q_R = \frac{3}{5}$ and $Q_L = \frac{8}{5}$ in figure 9(a), and $Q_R = \frac{1}{3}$ and $Q_L = \frac{4}{3}$ in figure 9(b). Again regions of high curvature are predicted.

5.4. *The case* $Q_R = \frac{2}{3}$

In the case $Q_R = \frac{2}{3}$ there are two different solutions in which the free surface $\psi = Q_R$ in (29) is smooth near $\theta = 0$, provided that jumps occur appropriately to allow the solution to develop a curtain. Specifically, the profile can be $h \sim 1 + \theta/\sqrt{6}$ near $\theta = 0$,

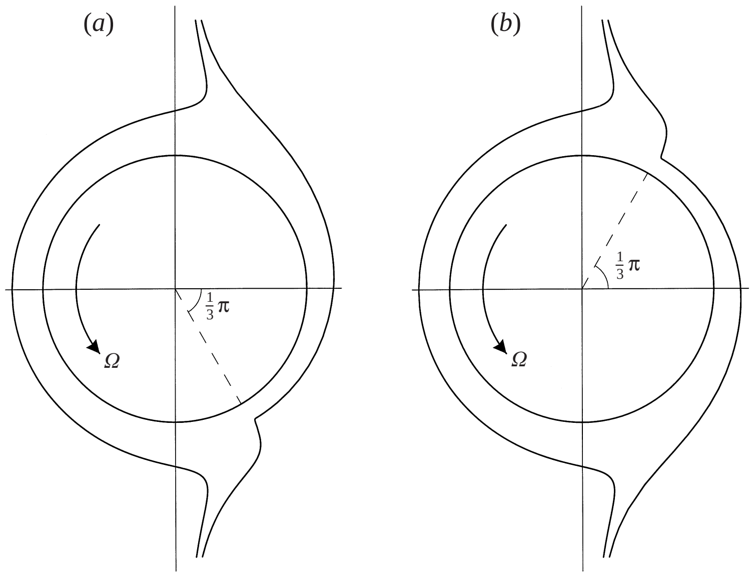

FIGURE 10. Two examples of jump-solution curtain flows when $Q_R = \frac{2}{3}$, drawn for the cases (*a*) $\theta_j = -\frac{1}{3}\pi$ and (*b*) $\theta_j = \frac{1}{3}\pi$, with $Q_L = \frac{5}{3}$, $Q_S = 1$ and $\delta = 0.02$ in both cases.

with a jump in $0 < \theta < \frac{1}{2}\pi$, or it can be $h \sim 1 - \theta/\sqrt{6}$ near $\theta = 0$, with a jump in $-\frac{1}{2}\pi < \theta < 0$. Examples of these cases (neither of which has top-to-bottom symmetry) are illustrated in figure 10, with $Q_R = \frac{2}{3}$, $Q_L = \frac{5}{3}$, $Q_S = 1$ and $\delta = 0.02$, and with jumps at $\theta_j = -\frac{1}{3}\pi$ and $\theta_j = \frac{1}{3}\pi$ in figures 10(*a*) and 10(*b*), respectively.

6. Films of finite thickness ($Q_S = 0$)

Finally we consider briefly films of the general type shown in figure 1(*a*), for which there is no supply flux ($Q_S = 0$) and the thickness h is finite everywhere. This sort of flow has been studied previously by, for example, Moffatt (1977), Preziosi & Joseph (1988) and Kelmanson (1995) .

6.1. *Full films*

Moffatt (1977) considered the case of a full film with no supply flux ($Q_S = 0$), with $Q(\theta)$ strictly constant (that is, independent of θ), and with $h(\theta)$ continuous (and so finite) for all θ. This requires $0 \leqslant Q \leqslant \frac{2}{3}$ (and $Q_R = Q_L = Q$), and the solution $h(\theta)$ is given by branch II above. The thickness of such a film decreases monotonically from a maximum h_{01} at $\theta = 0$ to a minimum h_1 at $\theta = \pi$. Again the solution is not fully determinate: for a given rotation rate there is a different solution for each value of Q (somewhat as in the drag-out problems mentioned in § 3).

The weight per unit length W of fluid on the cylinder (non-dimensionalized with $\rho g R L$) is

$$W = \int_0^{2\pi} h(\theta, Q)\,\mathrm{d}\theta, \tag{58}$$

from which it is found that W increases monotonically with Q. Moffatt was concerned with the question of maximizing the load W, and he showed that its maximum value is $W_{\max} = 4.44272\ldots$,† occurring when $Q = \frac{2}{3}$. In this state of maximum flux ($Q = \frac{2}{3}$) equation (29) gives $h \sim 1 - |\theta|/\sqrt{6}$ for $\theta \to 0$, showing that the free surface has a corner (see Moffatt 1977, footnote, p. 658, and Preziosi & Joseph 1988). This analysis of the maximum-supportable load has been extended by Kelmanson (1995) to include certain higher-order terms.

Note that there are no stagnation points in the flow in Moffatt's case, since in his solution $h(\theta) \leqslant \frac{3}{2}Q \leqslant 1$ for all θ, whereas the present analysis in § 5.1 shows that for a stagnation point to occur it is necessary that the film thickness satisfies $h \geqslant \sqrt{2}$ somewhere in the flow. We note also that in this case all the fluid in the film flows counter-clockwise, that is, in the same sense as the cylinder.

6.2. *Impossibility of partial films and jump solutions for coating flow*

Moffatt's (1977) full-film finite-thickness solution is valid for both coating flows and rimming flows. Johnson (1988) showed that, in addition, partial films and jump solutions are possible for rimming flows (see his figures 1, 6–8 and 11); and we have shown above that solutions of these types are also possible in curtain flows. It is of interest to note, however, that such solutions are not possible for coating flows when the film thickness is everywhere finite, that is, Moffatt's solution for coating flow cannot be generalized to include transitions of the type found by Johnson in rimming flows. As far as lubrication theory is concerned rimming flow and coating flow are the same, but for the $O(\delta)$ theory they are different: the free-surface slope h_θ predicted by (51) is always of the wrong sign to effect a smooth transition from one h-branch to another in coating flow, but not in rimming flow. For example, at an edge $\theta = \theta_e$ of a partial film, with $0 < \theta_e < \frac{1}{2}\pi$ (and $Q = 0$), equation (52) would imply $h_\theta > 0$ (since, as figure 3 shows, $f(h, 0) > \cos\theta_e$); this is impossible for a partial film in a coating flow. For rimming flow, on the other hand, the term involving δh_θ in (51) and (52) is preceded by a minus sign, and then transition solutions are indeed possible.

Physically we may reason that in rimming flow the upper edge of a partial film 'hangs from' the inside of the cylinder (and locally the flow is rather like that down the underside of an inclined plane), whereas in coating flow it would 'sit on' the outside of the cylinder (rather like flow down the topside of an inclined plane); it seems that the gravitational contribution to the pressure gradient that involves h_θ can be balanced in one case but not the other.

Note that for rimming flows, specifying values of Q and of θ_j or θ_e determines the weight W of fluid carried on the cylinder (and this may exceed Moffatt's 'maximum' $W_{\max}$ for a film with no jumps or edges). However, specifying Q and W does not determine fully the positions of any jumps or edges in the profile.

7. Summary and discussion

We have considered the steady two-dimensional flow that can ensue when a thin curtain of viscous fluid is incident on a cylinder rotating about its horizontal axis. Possible forms of the free surface and of streamline patterns (as predicted by a lubrication approximation) have been presented. The solution is physically sensible only in the part of fluid domain near to the cylinder: further away the solution

† In fact, Moffatt gave the slightly inaccurate value $W_{\max} = 4.428$. Also note that his footnote (p. 658) should have $\eta = \frac{3}{2} - \frac{1}{4}\sqrt{6}|\theta| + O(\theta^2)$ as $\theta \to 0$ (in his notation), corresponding to our equation (29).

becomes multi-valued, and must be interpreted with care. In particular a more detailed analysis of the curtain is needed, to understand the flow pattern therein, and to allow a matching with the thin film on the cylinder. The present theory predicts (perhaps surprisingly) that a curtain will buckle in a direction opposite to the rotation of the cylinder, and so will meet the cylinder right of centre, in the region $\theta_c < \theta < \frac{1}{2}\pi$. It might have been anticipated that a curtain would buckle in the direction of rotation, meeting the cylinder left of centre, in $\frac{1}{2}\pi < \theta < \pi$. However, we have found that this is a region that seems to be described well by the lubrication analysis, with no hint that anything other than what is shown in the figures will occur. Thus while there is certainly scope for improving the analysis to represent the curtain better (particularly the way that it impinges on the cylinder in the region right of centre, $\theta_c < \theta < \frac{1}{2}\pi$), it is not clear that such an improved theory could add anything beyond lubrication theory in the region left of centre ($\frac{1}{2}\pi < \theta < \pi$).

We have demonstrated several interesting features in the solutions, for example that stagnation points can occur in the flow, that solutions exist that do not have top-to-bottom symmetry, and that free-surface 'corners' are predicted for special parameter values. Also we have shown that in curtain flows the curtain generally takes a characteristic 'buckled' shape, and that in full-film curtain flows there is always some fluid that is 'trapped' near the rotating cylinder, circulating forever with the cylinder and never escaping as part of the curtain detaching at the bottom; in some cases all the fluid in the oncoming curtain goes round the left side of the cylinder, while in other cases some goes round the left side and some round the right side. In addition we have shown that neither finite-thickness full films involving jumps nor finite-thickness partial films can occur in the case of coating flows (though they can occur in rimming flows).

This paper has illustrated just some of the many possible interpretations of the lubrication solution; others are certainly feasible, but rather than elaborate further on these possibilities, it may be more profitable now to investigate what sort of solution would be chosen in practice. A way of tackling this theoretically may be to solve an initial value problem numerically for an unsteady flow (with the smoothing effect of surface tension included) to see what type of solution the system evolves towards; this was the method used by Wilson & Williams (1997) in their study of rimming flows, revealing the emergence of 'discontinuities' (regions of rapid variation) in the films. The question of the stability of these solutions is also of considerable interest.

Of course, the ultimate vindication of any theory can come only from experimental verification; we hope that an experimentalist will be stimulated to take up the challenge of testing some of our predictions!

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Appendix

The solutions $h_{\mathrm{I}}(\theta)$, $h_{\mathrm{II}}(\theta), \ldots, h_{\mathrm{V}}(\theta)$ of (19) and (20) on the branches I, II, ..., V (as in (23), (24) and (25)) may be written down explicitly. Perhaps the simplest forms are as follows.

With

$$K(\theta) := -\tfrac{3}{2}\mathrm{sgn}(\cos\theta)\, Q\sqrt{|\cos\theta|}, \tag{A 1}$$

we define functions $F_i(\theta)$ $(i = 1, \ldots, 4)$ by

$$F_1(\theta) = \frac{2}{\sqrt{\cos\theta}}\cos\left(\tfrac{1}{3}\cos^{-1}K(\theta)\right), \tag{A 2}$$

$$F_2(\theta) = \frac{2}{\sqrt{\cos\theta}}\cos\left(\frac{2\pi}{3} - \tfrac{1}{3}\cos^{-1}K(\theta)\right), \tag{A 3}$$

$$F_3(\theta) = \frac{2}{\sqrt{\cos\theta}}\cosh\left(\tfrac{1}{3}\cosh^{-1}K(\theta)\right), \tag{A 4}$$

$$F_4(\theta) = \frac{2}{\sqrt{|\cos\theta|}}\sinh\left(\tfrac{1}{3}\sinh^{-1}K(\theta)\right). \tag{A 5}$$

Then for $Q \leqslant 0$ we have

$$h_{\mathrm{I}} = \begin{cases} F_1(\theta) & \text{if } 0 \leqslant K \leqslant 1 \\ F_3(\theta) & \text{if } K > 1 \end{cases} \tag{A 6}$$

(defined only for $\cos\theta > 0$); for $0 < Q \leqslant \frac{2}{3}$ we have

$$h_{\mathrm{II}} = \begin{cases} F_2(\theta) & \text{if } \cos\theta > 0 \\ F_4(\theta) & \text{if } \cos\theta < 0, \end{cases} \tag{A 7}$$

and

$$h_{\mathrm{III}} = F_1(\theta) \tag{A 8}$$

(the latter defined only for $\cos\theta > 0$); and for $Q > \frac{2}{3}$ we have

$$h_{\mathrm{IV}} = \begin{cases} F_2(\theta) & \text{if } \theta_c \leqslant |\theta| \leqslant \frac{1}{2}\pi \\ F_4(\theta) & \text{if } \cos\theta < 0, \end{cases} \tag{A 9}$$

and

$$h_{\mathrm{V}} = F_1(\theta) \quad \text{if } \theta_c \leqslant |\theta| < \tfrac{1}{2}\pi, \tag{A 10}$$

with θ_c defined as in (22). Figure 4 shows these different branches schematically.

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